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Perturbation solution for small amplitude solitary waves in two-phase fluid flow of compacting media

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Abstract. A perturbation solution for small-amplitude solitary waves is derived for the third-order nonlinear partial differential equation due to Scott and Stevenson which describes the one-dimensional migration of melt through the Earth's mantle. The straightforward perturbation expansion breaks down and a coordinate stretching transformation is performed to render the perturbation expansion uniformly valid. The lowest-order perturbation solution has the same form as the single-soliton solution of the Korteweg–de Vries equation. The perturbation solution is derived to second order in implicit form. It is found to be extremely accurate when compared with known exact solutions for specific values of the exponents n and m. The zero- and first-order perturbation solutions are found to be accurate when $n = m$. The properties of small-amplitude solitary waves are investigated using the perturbation solution.

1. Introduction

The one-dimensional migration of melt upwards through the mantle of the Earth under the action of gravity can be described by the third-order nonlinear partial differential equation [1]

$$\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial z}\left[\phi^n\left(1 - \frac{\partial}{\partial z}\left(\frac{1}{\phi^m}\frac{\partial \phi}{\partial t}\right)\right)\right] = 0 \tag{1.1}$$

for the voidage, or volume fraction of melt, $\phi(z, t)$. Approximate large amplitude and exact rarefactive solitary wave solutions of equation (1.1) have been derived by several authors [1–5]. In a rarefactive solitary wave a small region of locally high voidage ascends through a background region of lower uniform voidage. In this paper we present a perturbation solution of equation (1.1) for small-amplitude rarefactive solitary waves.

The derivation of equation (1.1) has been performed by several authors, for both $m \neq 0$ [1, 5, 6] and for $m = 0$ [2, 3, 7]. Interest in equation (1.1) has continued [8–12] and recently Harris [13] performed a systematic search for conservation laws associated with the equation. The constant parameters n and m are the exponents in power laws relating the permeability of the medium, K, and the bulk and shear viscosities of the solid matrix, ξ and η, to the voidage:

$$K = K_0\phi^n \qquad \xi = \frac{\xi_0}{\phi^m} \qquad \eta = \frac{\eta_0}{\phi^m}. \tag{1.2}$$

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It has been suggested that the values of n and m in the ranges $2 \leqslant n \leqslant 5$ and $0 \leqslant m \leqslant 1$ are physically relevant to melt migration in the Earth's mantle [1, 3]. The voidage $\phi(z, t)$ in (1.1) is normalized by division using the constant background voidage ϕ_0 and in the derivation of (1.1) it is assumed that $\phi_0 \ll 1$. The dimensionless variable, z, is the vertical coordinate measured 'positive upwards' and scaled by division using the compaction length δ_c:

$$\delta_c = \left(\frac{K_0 \phi_0^{n-m} (\xi_0 + \frac{4}{3}\eta_0)}{\mu} \right)^{1/2} \tag{1.3}$$

where μ is the coefficient of shear viscosity of the melt. The dimensionless variable t is the time scaled by division using the characteristic time t_0:

$$t_0 = \left(\frac{\mu(\xi_0 + \frac{4}{3}\eta_0)}{K_0 \phi_0^{n+m-2}} \right)^{1/2} \frac{1}{g \Delta\rho} \tag{1.4}$$

where g is the acceleration due to gravity and $\Delta\rho = \rho_s - \rho_m > 0$ is the difference between the density of the solid matrix, ρ_s, and the density of the melt, ρ_m.

An outline of the paper is as follows. In section 2, general results required in the remainder of the paper are presented. In section 3, the perturbation solution for small-amplitude solitary waves is derived to second order in the perturbation parameter ε, where ε is the amplitude of the solitary wave. A stretching transformation is employed to render the perturbation expansion uniformly valid. In section 4, the accuracy of the perturbation solution for the speed of the solitary wave is examined and the properties of the speed of small-amplitude solitary waves are investigated. In section 5 the accuracy of the perturbation solution for the shape of the solitary wave is examined by comparing it with known exact solutions for specific values of n and m and the properties of small-amplitude solitary waves are investigated using the perturbation solution. Finally, the conclusions are summarized in section 6.

2. General results

In this section we review some general results which are required in section 3 [1, 3, 5].

We consider one-dimensional solitary wave solutions of (1.1) of the form

$$\phi(z, t) = \psi(\zeta) \qquad \zeta = z - ct \tag{2.1}$$

where the constant c is the dimensionless ascent speed of the solitary wave. Equation (1.1) becomes

$$c\frac{\mathrm{d}\psi}{\mathrm{d}\zeta} - \frac{\mathrm{d}}{\mathrm{d}\zeta}\left[\psi^n \left(1 + c\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\frac{1}{\psi^m}\frac{\mathrm{d}\psi}{\mathrm{d}\zeta}\right)\right)\right] = 0. \tag{2.2}$$

If we integrate (2.2) once with respect to ζ and then once with respect to ψ with the aid of the identity [1]

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\frac{1}{\psi^m}\frac{\mathrm{d}\psi}{\mathrm{d}\zeta}\right) = \frac{1}{2}\psi^m \frac{\mathrm{d}}{\mathrm{d}\psi}\left(\frac{1}{\psi^{2m}}\left(\frac{\mathrm{d}\psi}{\mathrm{d}\zeta}\right)^2\right) \tag{2.3}$$

we obtain

$$c\left(\frac{\mathrm{d}\psi}{\mathrm{d}\zeta}\right)^2 = f(\psi) \tag{2.4}$$

where

$$f(\psi) = 2\psi^{2m}\left(\alpha \int^\psi \frac{\mathrm{d}x}{x^{n+m}} + c\int^\psi \frac{\mathrm{d}x}{x^{n+m-1}} - \int^\psi \frac{\mathrm{d}x}{x^m} + \beta\right) \tag{2.5}$$

and α and β are constants. The background state is $\psi = 1$. Let the maximum value of the voidage be $\psi = \Psi > 1$ so that the amplitude of the solitary wave is $\Psi - 1$. The three constants, α, β and c in (2.5) are obtained by imposing the following three boundary conditions for a rarefactive solitary wave:

$$\psi = 1 \qquad \frac{\mathrm{d}\psi}{\mathrm{d}\zeta} = 0 \qquad \frac{\mathrm{d}^2\psi}{\mathrm{d}\zeta^2} = 0 \tag{2.6}$$

$$\psi = \Psi > 1 \qquad \frac{\mathrm{d}\psi}{\mathrm{d}\zeta} = 0. \tag{2.7}$$

By using (2.3) with $m = 0$ and (2.4), the three boundary conditions given by (2.6) and (2.7) can be written in terms of $f(\psi)$ as

$$f(1) = 0 \qquad \frac{\mathrm{d}f}{\mathrm{d}\psi}(1) = 0 \qquad f(\Psi) = 0 \quad \Psi > 1. \tag{2.8}$$

It can be shown [5] that if the three boundary conditions in (2.8) are imposed on (2.5) then $(\frac{\mathrm{d}\psi}{\mathrm{d}\zeta})^2 < 0$ for $1 < \psi < \Psi$ when $0 < n \leqslant 1$ and that $c = 0$ and $\frac{\mathrm{d}\psi}{\mathrm{d}\zeta}$ is undetermined when $n = 0$. Therefore, rarefactive solitary wave solutions satisfying (2.8) do not exist when $0 \leqslant n \leqslant 1$. It can be further shown that if $n > 1$ then $(\frac{\mathrm{d}\psi}{\mathrm{d}\zeta})^2 > 0$ for $1 < \psi < \Psi$ and hence rarefactive solitary wave solutions exist. In the following we therefore only consider the range $n > 1$.

There are four cases: a general case and three special cases. For the general case in which $n + m \neq 1$, $n + m \neq 2$ and $m \neq 1$, it can be verified that [5]

$$c = \frac{(n+m-2)[n\Psi^{n+m-1} - (n+m-1)\Psi^n + m - 1]}{(m-1)[\Psi^{n+m-1} - (n+m-1)\Psi + n + m - 2]} \tag{2.9}$$

$$\begin{aligned}\left(\frac{\mathrm{d}\psi}{\mathrm{d}\zeta}\right)^2 = {} & \frac{2}{(n+m-2)[n\Psi^{n+m-1} - (n+m-1)\Psi^n + m - 1]} \\ & [+((n-1)\Psi^{n+m-1} - (n+m-2)\Psi^n + (m-1)\Psi)\psi^{m-n+1} \\ & -(n\Psi^{n+m-1} - (n+m-1)\Psi^n + m - 1)\psi^{m-n+2} \\ & +(\Psi^{n+m-1} - (n+m-1)\Psi + n + m - 2)\psi^{m+1} - (\Psi^n - n\Psi + n - 1)\psi^{2m}].\end{aligned} \tag{2.10}$$

The three special cases are $n + m = 1$, $n + m = 2$ and $m = 1$. The perturbation solution which we derive for the general case remains valid when we put $n+m = 1$, $n+m = 2$ or $m = 1$ and it can be verified by direct calculation that the perturbation solutions for the three special cases are exactly the same as are obtained by putting $n + m = 1$, $n + m = 2$ or $m = 1$ in the perturbation solution for the general case. The singular factors $n+m-1$, $n+m-2$ and $m-1$ which occur at intermediate stages in the analysis in the general case cancel before the end of the calculation and do not occur in the final perturbation solution. Logarithms which occur at intermediate stages in the analysis of the special cases do not occur in the final perturbation solutions for the special cases. Therefore, we do not consider the special cases here. The analysis is the same as the general case and the perturbation solution for the general case is valid for all values of n and m.

3. Perturbation solution

We consider small-amplitude solitary waves. Let ε denote the amplitude of the solitary wave so that $\Psi = 1 + \varepsilon$ where $0 < \varepsilon < 1$. We take ε to be the perturbation parameter.

The expansion of (2.9) in powers of ε to third order in ε is

$$c = n[1 + \tfrac{1}{3}(n-1)\varepsilon + \tfrac{1}{36}(n-1)(2n-m-6)\varepsilon^2 + \tfrac{1}{540}(n-1)(2n^2-8nm-m^2-24n+21m+54)\varepsilon^3 + \mathrm{O}(\varepsilon^4)] \qquad (3.1)$$

as $\varepsilon \to 0$. The factors $(n+m-2)$ and $(m-1)$ which occur in (2.9) do not occur in (3.1).

Next consider the perturbation solution for the solitary wave ψ. Many of the exact solutions which have been derived are expressed in the implicit form, $\zeta = \zeta(\psi)$, [2, 3, 5]. We therefore look for a perturbation solution in implicit form and treat ψ as the independent variable and ζ as the dependent variable. Now, a straightforward perturbation expansion of the form

$$\zeta(\psi;\varepsilon) = \frac{1}{\varepsilon^\alpha}(\zeta_0(\psi) + \varepsilon\zeta_1(\psi) + \varepsilon^2\zeta_2(\psi) + \mathrm{O}(\varepsilon^3)) \qquad (3.2)$$

as $\varepsilon \to 0$, where ψ is kept fixed in the limiting process and the exponent α is to be determined during the analysis, breaks down. The reason why (3.2) breaks down is due to the factor $(\Psi - \psi)$ in $f(\psi)$ which occurs because of the boundary condition $f(\Psi) = 0$ given by (2.8). Then $\frac{d\psi}{d\zeta}$ has the factor $(\Psi-\psi)^{1/2}$ which may be written as $(1-\psi+\varepsilon)^{1/2}$. The lowest-order term in the straightforward perturbation expansion of $\frac{d\psi}{d\zeta}$, which is obtained by putting $\varepsilon = 0$, therefore has the factor $(1-\psi)^{1/2}$ which is imaginary because $\psi > 1$. The non-existence of a straightforward perturbation solution of the form (3.2) is due to the sharp change in the dependent variable ζ, from either $\zeta = +\infty$ or $-\infty$ to $\zeta = 0$ in the domain $1 \leqslant \psi \leqslant 1+\varepsilon$ of the independent variable ψ. The sharp-change region is $\psi - 1 = \mathrm{O}(\varepsilon)$. A sharp change is characterized by a magnified scale. We therfore magnify the sharp change by the stretching transformation

$$u = \frac{\psi - 1}{\varepsilon}. \qquad (3.3)$$

The domain of the variable u is $0 \leqslant u \leqslant 1$. We look for a perturbation solution of the form

$$\zeta(u;\varepsilon) = \frac{1}{\varepsilon^\alpha}(\zeta_0(u) + \varepsilon\zeta_1(u) + \varepsilon^2\zeta_2(u) + \mathrm{O}(\varepsilon^3)) \qquad (3.4)$$

as $\varepsilon \to 0$, where u is kept fixed in the limiting process and the exponent α is to be determined during the analyis by balancing the dominant terms. The problem is a singular perturbation problem.

We expand (2.10) to third order in ε by substituting $\psi = 1+\varepsilon u$ and $\Psi = 1+\varepsilon$. This gives

$$\left(\frac{du}{d\zeta}\right)^2 = \frac{\varepsilon(n-1)u^2(1-u)}{3}\Bigg[1 - \frac{\varepsilon}{12}\{3(2n-5m+2)u + 2n + m + 2\} + \frac{\varepsilon^2}{189}\{9(3n^2-12nm+16m^2+9n-26m+6)u^2 + 3(4n^2-11nm-7m^2+12n-8m+8)u + 2n^2+2nm-m^2+16n+11m+14\} + \mathrm{O}(\varepsilon^3)\Bigg] \qquad (3.5)$$

as $\varepsilon \to 0$. Intermediate results in the derivation of (3.5) are given in appendix A. We observe that $n-1$ is a factor on the right-hand side of equation (3.5). This clearly illustrates the general result that for solitary wave solutions to exist which satisfy the boundary conditions (2.6) and (2.7), or equivalently (2.8), it is necessary that $n > 1$. We also see that the factors $n+m-1$, $n+m-2$ and $m-1$ which occur at intermediate stages in the analysis do not occur in expansion (3.5).

It follows directly from (3.5) and expansion (3.4) for $\zeta(u;\varepsilon)$ that

$$\frac{1}{\varepsilon^{\alpha}}(\mathrm{d}\zeta_0 + \varepsilon\,\mathrm{d}\zeta_1 + \varepsilon^2\,\mathrm{d}\zeta_2 + \mathrm{O}(\varepsilon^3))$$
$$= \pm\frac{1}{\varepsilon^{1/2}}\left(\frac{3}{n-1}\right)^{1/2}\left[\frac{1}{u(1-u)^{1/2}} + \varepsilon\left(\frac{A(n,m)}{2(1-u)^{1/2}} + \frac{B(n,m)}{u(1-u)^{1/2}}\right)\right.$$
$$\left. +\varepsilon^2\left(\frac{3}{2}D(n,m)(1-u)^{1/2} + \frac{E(n,m)}{2(1-u)^{1/2}} + \frac{F(n,m)}{u(1-u)^{1/2}}\right) + \mathrm{O}(\varepsilon^3)\right]\mathrm{d}u \qquad (3.6)$$

where

$$A(n,m) = \tfrac{1}{4}(2n - 5m + 2) \qquad (3.7)$$
$$B(n,m) = \tfrac{1}{24}(2n + m + 2) \qquad (3.8)$$
$$D(n,m) = \tfrac{1}{960}(-12n^2 + 108nm - 119m^2 + 24n - 116m + 36) \qquad (3.9)$$
$$E(n,m) = \tfrac{1}{960}(92n^2 - 388nm + 319m^2 - 24n + 236m - 116) \qquad (3.10)$$
$$F(n,m) = \tfrac{1}{5760}(28n^2 + 28nm + 31m^2 - 136n - 116m - 164). \qquad (3.11)$$

By balancing the dominant terms on the right- and left-hand sides of (3.6) it follows that $\alpha = \frac{1}{2}$. Equation (3.6) becomes

$$\mathrm{d}\zeta_0 + \varepsilon\mathrm{d}\zeta_1 + \varepsilon^2\mathrm{d}\zeta_2 + \mathrm{O}(\varepsilon^3) = \pm\left(\frac{3}{n-1}\right)^{1/2}\left[\frac{1}{u(1-u)^{1/2}} + \varepsilon\left(\frac{A(n,m)}{2(1-u)^{1/2}} + \frac{B(n,m)}{u(1-u)^{1/2}}\right)\right.$$
$$\left. +\varepsilon^2\left(\frac{3}{2}D(n,m)(1-u)^{1/2} + \frac{E(n,m)}{2(1-u)^{1/2}} + \frac{F(n,m)}{u(1-u)^{1/2}}\right) + \mathrm{O}(\varepsilon^3)\right]\mathrm{d}u. \qquad (3.12)$$

We choose $\zeta = 0$ at $u = 1$ which corresponds to $\psi = \Psi$. Thus

$$u = 1 \qquad \zeta_n(1) = 0 \qquad n \geqslant 0. \qquad (3.13)$$

Equation (3.12) is solved, subject to the initial conditions (3.13), by equating the coefficients of like powers of ε.

3.1. Zero order in ε

The zero order in ε terms in (3.12) are

$$\mathrm{d}\zeta_0 = \pm\left(\frac{3}{n-1}\right)^{1/2}\frac{\mathrm{d}u}{u(1-u)^{1/2}}. \qquad (3.14)$$

We obtain the zero-order solution in both explicit and implicit form.

First, consider the solution in explicit form. If we let $u = \mathrm{sech}^2\,\theta$ and from (3.13) impose the initial condition $\zeta_0 = 0$ at $\theta = 0$, then we obtain

$$u = \mathrm{sech}^2\left(\frac{1}{2}\left(\frac{n-1}{3}\right)^{1/2}\zeta_0\right). \qquad (3.15)$$

Since $\psi = 1 + \varepsilon u$ and, to lowest order in ε, $\zeta_0 = \varepsilon^{1/2}\zeta$, where $\varepsilon = \Psi - 1$, equation (3.15) becomes

$$\psi = 1 + (\Psi - 1)\,\mathrm{sech}^2\left[\frac{1}{2}\left(\frac{(n-1)(\Psi-1)}{3}\right)^{1/2}\zeta\right]. \qquad (3.16)$$

The approximate solution (3.16) has the same sech^2 form as the single-soliton solution of the Korteweg–de Vries equation [14].

Next consider the solution in implicit form. By letting $u = 1 - w^2$ it can be verified that

$$\int \frac{\mathrm{d}u}{u(1-u)^{1/2}} = -\ln\left(\frac{1+(1-u)^{1/2}}{1-(1-u)^{1/2}}\right) + \text{constant}. \tag{3.17}$$

Thus using (3.17) and the initial condition, $\zeta_0 = 0$ at $u = 1$, it follows from (3.14) that

$$\zeta_0(u) = \mp\left(\frac{3}{n-1}\right)^{1/2} \ln\left(\frac{1+(1+u)^{1/2}}{1-(1-u)^{1/2}}\right). \tag{3.18}$$

Since $\zeta_0 = \varepsilon^{1/2}\zeta$ and $\psi = 1 + \varepsilon u$, where $\varepsilon = \Psi - 1$, we obtain from (3.18) the implicit solution to lowest order in ε:

$$\zeta = \mp\left(\frac{3}{(n-1)(\Psi-1)}\right)^{1/2} \ln\left(\frac{(\Psi-1)^{1/2}+(\psi-1)^{1/2}}{(\Psi-1)^{1/2}-(\psi-1)^{1/2}}\right). \tag{3.19}$$

3.2. *First order in ε*

By equating the coefficients of ε in (3.12) we obtain

$$\mathrm{d}\zeta_1 = \pm\left(\frac{3}{n-1}\right)^{1/2}\left[\frac{A(n,m)}{2(1-u)^{1/2}} + \frac{B(n,m)}{u(1-u)^{1/2}}\right]\mathrm{d}u. \tag{3.20}$$

We integrate (3.20) with the aid of (3.17) and impose the initial condition (3.13) that $\zeta_1 = 0$ at $u = 1$. This gives

$$\zeta_1(u) = \mp\left(\frac{3}{n-1}\right)^{1/2}\left[A(n,m)(1-u)^{1/2} + B(n,m)\ln\left(\frac{1+(1-u)^{1/2}}{1-(1-u)^{1/2}}\right)\right]. \tag{3.21}$$

3.3. *Second order in ε*

By equating the coefficients of ε^2 in (3.12) we have

$$\mathrm{d}\zeta_2 = \pm\left(\frac{3}{n-1}\right)^{1/2}\left[\frac{3}{2}D(n,m)(1-u)^{1/2} + \frac{E(n,m)}{2(1-u)^{1/2}} + \frac{F(n,m)}{u(1-u)^{1/2}}\right]\mathrm{d}u. \tag{3.22}$$

In order to integrate (3.22), equation (3.17) is used again. By imposing the initial condition from (3.13) that $\zeta_2 = 0$ at $u = 1$, we obtain

$$\zeta_2(u) = \mp\left(\frac{3}{n-1}\right)^{1/2} \times\left[D(n,m)(1-u)^{3/2} + E(n,m)(1-u)^{1/2} + F(n,m)\ln\left(\frac{1+(1-u)^{1/2}}{1-(1-u)^{1/2}}\right)\right]. \tag{3.23}$$

3.4. *Solution correct to order ε^2*

The perturbation solution correct to second order in ε is obtained by substituting (3.18), (3.21) and (3.23) into expansion (3.4) with $\alpha = \frac{1}{2}$:

$$\begin{aligned}\zeta = \mp\left(\frac{3}{(n-1)\varepsilon}\right)^{1/2}&\left[\ln\left(\frac{1+(1-u)^{1/2}}{1-(1-u)^{1/2}}\right)\right.\\ &+\varepsilon\left(A(n,m)(1-u)^{1/2} + B(n,m)\ln\left(\frac{1+(1-u)^{1/2}}{1-(1-u)^{1/2}}\right)\right)\\ &+\varepsilon^2\left(D(n,m)(1-u)^{3/2} + E(n,m)(1-u)^{1/2}\right.\\ &\left.\left.+F(n,m)\ln\left(\frac{1+(1-u)^{1/2}}{1-(1-u)^{1/2}}\right)\right) + \mathrm{O}(\varepsilon^3)\right]\end{aligned} \tag{3.24}$$

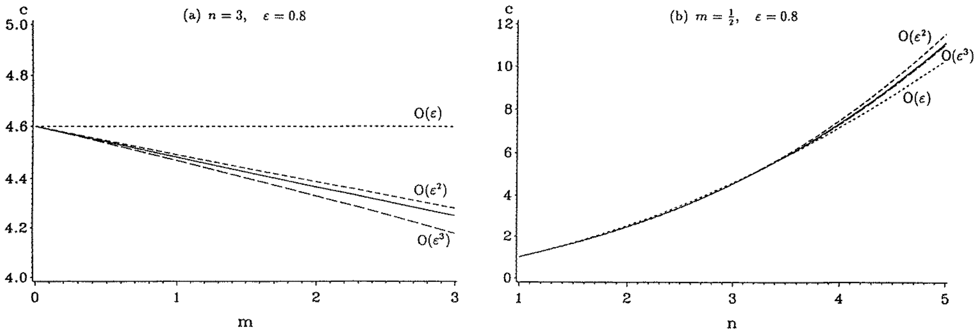

Figure 1. Comparison of the perturbation solution (3.1) and the exact solution (2.9) for c with $\varepsilon = 0.8$ and (*a*) $n = 3$, $0 \leqslant m \leqslant 3$ (*b*) $m = \frac{1}{2}$, $1 < n \leqslant 5$. Perturbation solution to first order (- - - -), second order (– – –), third order (— — —) and exact solution (——).

as $\varepsilon \to 0$, where

$$\psi = 1 + \varepsilon u. \tag{3.25}$$

We can substitute $u = \frac{1}{\varepsilon}(\psi - 1)$ into (3.24) to obtain an implicit solution of the form $\zeta = \zeta(\psi)$. Alternatively, we can leave the solution in the parametric form, (3.24) and (3.25), with u as a parameter, where $0 \leqslant u \leqslant 1$. The parametric form is particularly useful for plotting graphs of the solitary wave.

The perturbation solution for the special cases $n + m = 1$, $n + m = 2$ and $m = 1$ can be calculated, as outlined above, for the general case. It can be verified that the perturbation solutions derived for the special cases are exactly the same as are obtained by putting $m = 1 - n$, $m = 2 - n$ and $m = 1$ in the general solution (3.24). The special case $n + m = 1$ is not physical if $m \geqslant 0$ because then $n \leqslant 1$.

The perturbation solution, (3.24) and (3.25), is discussed in section 5.

4. Perturbation solution for the speed of the solitary wave

The accuracy of the perturbation solution for c as m varies for fixed n is examined in figure 1(*a*) where the perturbation solution (3.1) and the exact solution (2.9) for $n = 3$ and $0 \leqslant m \leqslant 3$ are compared when $\varepsilon = 0.8$. The perturbation solution becomes less accurate as m increases and the perturbation solution to $\mathrm{O}(\varepsilon^2)$ is slightly more accurate than the perturbation solution to $\mathrm{O}(\varepsilon^3)$. The accuracy of the perturbation solution for c as n varies for fixed m is investigated in figure 1(*b*) where the perturbation and exact solutions, (3.1) and (2.9), are compared for $m = \frac{1}{2}$ and $1 \leqslant n \leqslant 5$ when $\varepsilon = 0.8$. The perturbation solution becomes less accurate as n increases but the perturbation solution to $\mathrm{O}(\varepsilon^3)$ is more accurate than the perturbation solution to $\mathrm{O}(\varepsilon^2)$. As more terms are included in the perturbation solution it alternately underestimates and overestimates the exact solution.

The dependence of c on n has been fully investigated using the exact theory [5] and it can be shown that c is an increasing function of n for $n > 1$, as illustrated in figure 1(*b*). The dependence of c on m can be investigated using the perturbation solution (3.1). From (3.1),

$$\frac{\partial c}{\partial m} = -\frac{1}{540} n(n-1)[15 + (8n + 2m - 21)\varepsilon]\varepsilon^2 + \mathrm{O}(\varepsilon^4) \tag{4.1}$$

as $\varepsilon \to 0$. Suppose that $m \geqslant 0$, which is physically relevant to melt migration in the Earth's mantle and that $0 \leqslant \varepsilon < 1$ which is necessary for the perturbation solution to apply. Also, for

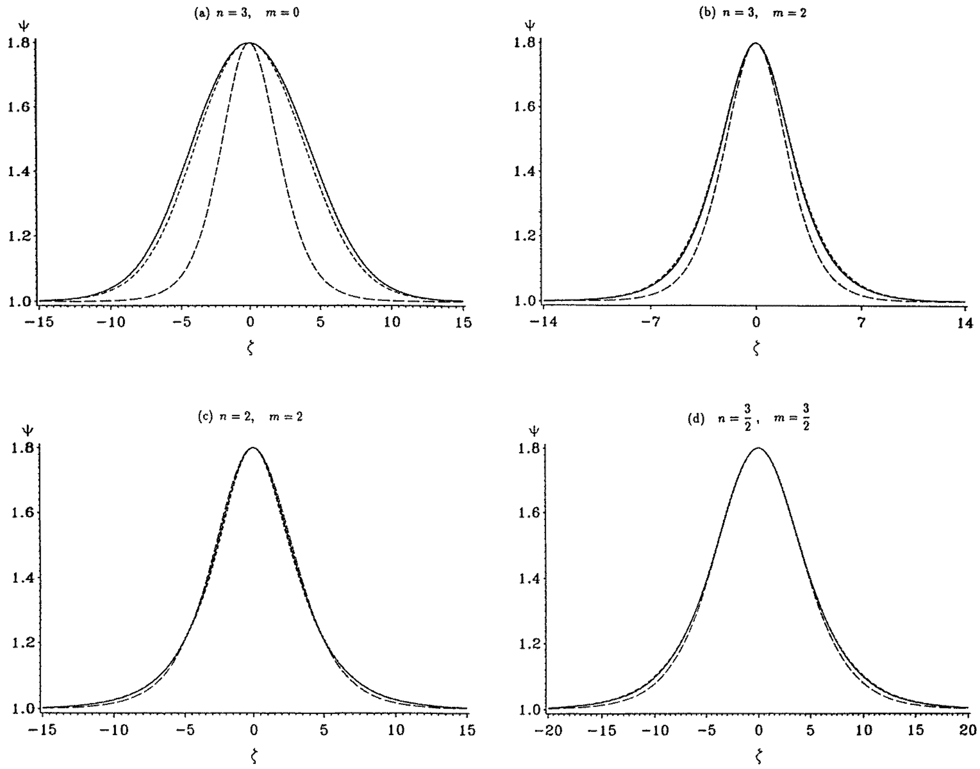

Figure 2. Comparison of the perturbation solution, (3.24) and (3.25), for $\varepsilon = 0.8$ with the exact solutions: (*a*) $n = 3$, $m = 0$; (*b*) $n = 3$, $m = 2$; (*c*) $n = m = 2$; (*d*) $n = m = \frac{3}{2}$. Perturbation solution to zero order (— — —), first order (– – –), second order (- - - -) and exact solution (——). The graphs of the second-order perturbation solution and the exact solutions overlap.

the existence of a solution, it is necessary that $n > 1$. Then, if terms of $O(\varepsilon^4)$ are neglected,

$$\frac{\partial c}{\partial m} < -\frac{1}{270} n(n-1)(m+1)\varepsilon^3 < 0. \tag{4.2}$$

Thus, for small-amplitude solitary waves correct to $O(\varepsilon^3)$, c is a decreasing function of the exponent m for $m \geqslant 0$, as illustrated in figure 1(*a*).

The speed, c, is dimensionless and is scaled with respect to the velocity of the background melt which from (1.3) and (1.4) is given by

$$\frac{\delta c}{t_0} = \left(\frac{K_0 g \Delta\rho}{\mu}\right) \phi_0^{n-1}. \tag{4.3}$$

The actual speed, v, of the solitary wave is

$$v = \left(\frac{K_0 g \Delta\rho}{\mu}\right) \phi_0^{n-1} c \tag{4.4}$$

which has a different dependence on n from c but the same dependence on m. Thus for small-amplitude solitary waves correct to $O(\varepsilon^3)$, v is a decreasing function of the exponent m for $m \geqslant 0$.

5. Perturbation solution for the solitary wave

The exact solutions for $n = 3$ and $m = 0$, $n = 3$ and $m = 2$, $n = m = 2$ and $n = m = \frac{3}{2}$ are listed in appendix B. In figure 2 the perturbation solution in parametric form, given by (3.24) and (3.25), is compared with the exact solutions when $\varepsilon = 0.8$. For all cases the graph of the perturbation solution to second order in ε overlaps the exact solution which indicates that the perturbation solution to second order in ε is a good approximation for small-amplitude solitary waves. We also see that when $m = n$ the perturbation solution is particularly accurate and even the zero-order solution (3.16) is close to the exact solution.

In order to use the perturbation solution to investigate how the properties of small amplitude solitary waves depend on n and m, we must express the solution in terms of characteristic quantities that are independent of n and m. Thus, instead of scaling ζ by δ_c defined by (1.3), we use the characteristic length δ_c' defined by

$$\delta_c' = \left(\frac{K_0(\xi_0 + \frac{4}{3}\eta_0)}{\mu} \right)^{1/2}. \tag{5.1}$$

The perturbation solution depends explicitly on the background voidage ϕ_0 where $\phi_0 \ll 1$.

First, consider the lowest-order perturbation solution given by (3.16). When ζ is scaled by δ_c', (3.16) may be written as

$$\psi = 1 + (\Psi - 1)\,\mathrm{sech}^2\left(\frac{z - ct}{L}\right) \tag{5.2}$$

where L, which is a measure of the width of the solitary wave to this approximation, is

$$L = 2\left(\frac{3\phi_0^{n-m}}{(n-1)(\Psi - 1)}\right)^{1/2}. \tag{5.3}$$

Equation (5.2) has the same form as the single-soliton solution of the Korteweg–de Vries equation [14]. The width, L, is inversely proportional to the square root of the amplitude as with the solution of the Korteweg–de Vries equation. Larger-amplitude solitary waves are therefore narrower in width. This compares with the result for large-amplitude solitary waves for which it can be shown, using the large-amplitude approximation, that larger-amplitude solitary waves are narrower in width if $m > 1$ but broader in width if $0 \leqslant m \leqslant 1$ [5]. It follows directly from (5.3) that

$$\frac{\partial L}{\partial n} = \frac{1}{2}\left(\ln\phi_0 - \frac{1}{n-1}\right) L < 0 \tag{5.4}$$

$$\frac{\partial L}{\partial m} = -\frac{1}{2}(\ln\phi_0) L > 0. \tag{5.5}$$

Thus, to the lowest-order approximation, L decreases as n increases and L increases as m increases. If terms to first order in ε are retained then from (3.1),

$$c = n(1 + \tfrac{1}{3}(n-1)(\Psi - 1)). \tag{5.6}$$

The speed c increases linearly with the amplitude, $\Psi - 1$, as for the single-soliton solution of the Korteweg–de Vries equation [14]. Thus, in the small-amplitude approximation, larger-amplitude solitary waves travel faster. The fact that larger-amplitude solitary waves travel faster may be proved without approximation for waves of any amplitude [5].

Now consider the contributions of higher-order terms in the perturbation solution. Let W denote the width of the solitary wave at half its height. Then W is twice the value of ζ

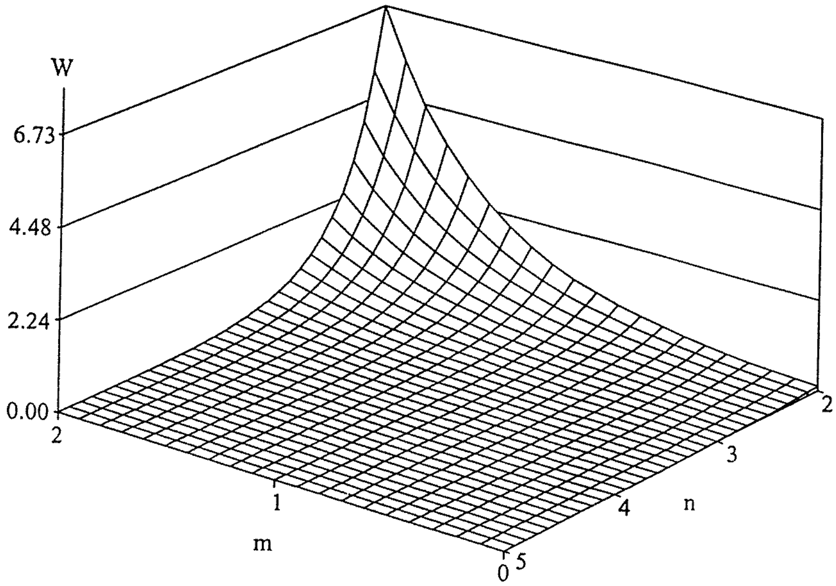

Figure 3. The width, W, of the solitary wave at half its height, given by (5.7), plotted against n and m for $\phi_0 = 0.01$ and $\varepsilon = 0.8$. The characteristic length is δ_c' defined by (5.1).

evaluated at $u = \frac{1}{2}$. If we scale ζ by δ_c' instead of by δ_c then from (3.24),

$$W = 2\left(\frac{3\phi_0^{n-m}}{(n-1)\varepsilon}\right)^{1/2}\left[\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) + \varepsilon\left(\frac{1}{\sqrt{2}}\,A(n,m) + B(n,m)\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)\right)\right.$$
$$\left. +\varepsilon^2\left(\frac{1}{2\sqrt{2}}D(n,m) + \frac{1}{\sqrt{2}}E(n,m) + F(n,m)\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)\right) + \mathrm{O}(\varepsilon^3)\right] \quad (5.7)$$

as $\varepsilon \to 0$. In figure 3, W is plotted against n and m for $\phi_0 = 0.01$ and $\varepsilon = 0.8$. We see from figure 3 that when $\phi_0 = 0.01$, W is an increasing function of m and a decreasing function of n which is consistent with the conclusions drawn from the lowest-order perturbation solution. A similar dependence of the width on n and m when $\phi_0 = 0.01$ was found for the large-amplitude approximation [5].

6. Conclusions

The exact solutions of equation (1.1) which have been derived so far apply for specific values of n and m. The perturbation solution, (3.24) and (3.25), applies for all values of $n(n > 1)$ and m. It can therefore be used to investigate how the properties of small-amplitude solitary waves depend on n and m. Equation (3.24) clearly shows that $n > 1$ is a necessary condition for the existence of solitary wave solutions satisfying the boundary conditions (2.6) and (2.7) [5]. The parametric form, (3.24) and (3.25), also provides a convenient way to calculate the width and to plot graphs of the solitary wave.

The comparison of the perturbation solution with the exact solutions showed that the perturbation solution to order ε^2 is a very good approximate small-amplitude solitary wave solution. In all cases considered the graphs of the perturbation solution to second order in ε and the exact solutions overlapped. The accuracy of the perturbation solutions to zero order and first order in ε improved as m and n became equal and were quite accurate for the two cases of $m = n$ considered. The values of n and m in the exact solutions used to test the perturbation solution were in the ranges $\frac{3}{2} \leqslant n \leqslant 3$ and $0 \leqslant m \leqslant 2$. The value, $\varepsilon = 0.8$, which was used is comparatively large for a perturbation parameter and provided a good test.

The lowest-order perturbation solution has the same sech^2 form and similar properties to the single-soliton solution of the Korteweg–de Vries equation. To this approximation the width of the solitary wave is inversely proportional to the square root of the amplitude and the speed of the solitary wave increases linearly with the amplitude.

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Appendix A. Expansions in powers of ε

Consider the general case in which $n + m \neq 1$, $n + m \neq 2$ and $m \neq 1$. We present some perturbation expansions to fifth order in ε used in the derivation of equation (3.5). Define

$$\begin{aligned} S(n, m; \varepsilon) = {}& \tfrac{1}{3}\varepsilon - \tfrac{1}{36}(2n + m + 2)\varepsilon^2 + \tfrac{1}{540}(2n^2 + 2nm - m^2 + 16n + 11m + 14)\varepsilon^3 \\ & + \tfrac{1}{6480}(2n^3 + 3n^2m + 3nm^2 + m^3 - 24n^2 - 27nm \\ & + 12m^2 - 126n - 102m - 100)\varepsilon^4 \\ & - \tfrac{1}{136\,080}(10n^4 + 20n^3m + 6n^2m^2 - 4nm^3 - 5m^4 \\ & + 58n^3 + 105n^2m + 111nm^2 + 50m^3 \\ & - 462n^2 - 561nm + 204m^2 - 1922n - 1726m - 1412)\varepsilon^5. \end{aligned} \tag{A.1}$$

The expansions to fifth order in ε of the ratios of functions of $\Psi = 1 + \varepsilon$ which occur in (2.10) are

$$\frac{(n-1)\Psi^{n+m-1} - (n+m-2)\Psi^n + (m-1)\Psi}{n\Psi^{n+m-1} - (n+m-1)\Psi^n + (m-1)} = \frac{(n-1)(n+m-2)}{n(n+m-1)}[1 + S(n, m; \varepsilon) + \mathrm{O}(\varepsilon^6)] \tag{A.2}$$

$$\frac{\Psi^{n+m-1} - (n+m-1)\Psi + n + m - 2}{n\Psi^{n+m-1} - (n+m-1)\Psi^n + m - 1} = \frac{(n+m-2)}{n(m-1)}[1 - (n-1)S(n, m; \varepsilon) + \mathrm{O}(\varepsilon^6)] \tag{A.3}$$

$$\frac{\Psi^n - n\Psi + n - 1}{n\Psi^{n+m-1} - (n+m-1)\Psi^n + m - 1} = \frac{(n-1)}{(n+m-1)(m-1)}[1 - (n+m-2)S(n, m; \varepsilon) + \mathrm{O}(\varepsilon^6)] \tag{A.4}$$

as $\varepsilon \to 0$. We see that the three equations, (A.2) to (A.4), can be expressed in terms of the same expansion $S(n, m; \varepsilon)$.

Appendix B. Exact solutions

We list the exact solutions which are used to check the accuracy of the perturbation solution [2, 3, 5].

(i) $n = 3$ and $m = 0$

$$\zeta = \mp\left(\Psi + \frac{1}{2}\right)^{1/2}\left[-2(\Psi - \psi)^{1/2} + \frac{1}{(\Psi - 1)^{1/2}}\ln\left(\frac{(\Psi - 1)^{1/2} - (\Psi - \psi)^{1/2}}{(\Psi - 1)^{1/2} + (\Psi - \psi)^{1/2}}\right)\right]. \tag{B.1}$$

(ii) $n = 3$ and $m = 2$

$$\psi = \Psi\left(\frac{1 - A\tanh^2(D\zeta)}{1 + B\tanh^2(D\zeta)}\right) \tag{B.2}$$

where

$$A = \frac{(2\Psi + 1)(\Psi - 1)}{3\Psi(\Psi + 1)} \qquad B = \frac{(\Psi + 2)(\Psi - 1)}{3(\Psi + 1)}$$
$$D = \left(\frac{(\Psi + 1)(\Psi - 1)}{2(3\Psi^2 + 2\Psi + 1)}\right)^{1/2}. \tag{B.3}$$

(iii) $n = 2$ and $m = 2$

$$\psi = \frac{\Psi}{1 + (\Psi - 1)\tanh^2\left(\left(\frac{(\Psi - 1)}{4(2\Psi + 1)}\right)^{1/2}\zeta\right)}. \tag{B.4}$$

(iv) $n = \frac{3}{2}$ and $m = \frac{3}{2}$

$$\psi = \Psi\left(\frac{1 - A\tanh^2(D\zeta)}{1 + B\tanh^2(D\zeta)}\right)^2 \tag{B.5}$$

where

$$A = \frac{\Psi^{1/2} - 1}{3\Psi^{1/2} + 1} \qquad B = \frac{(2\Psi^{1/2} + 1)(\Psi^{1/2} - 1)}{3\Psi^{1/2} + 1}$$
$$D = \left(\frac{(\Psi^{1/2} - 1)(3\Psi^{1/2} + 1)}{8(3\Psi + 2\Psi^{1/2} + 1)}\right)^{1/2}. \tag{B.6}$$

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